ANALYTICAL SOLUTIONS FOR UNSTEADY FLOW PROBLEMS OF MAXWELL FLUID IN A POROUS MEDIAM

[*1]I. Siddique, [1]Abdullah Y. Al Hossain, [2]M. B. Riaz

[1]Department of Mathematics, Jazan University, Jazan, P. O. Box 2097, Saudi Arabia.

[2]Department of Mathematics, University of Management and Technology, Lahore, Pakistan.

imransmsrazi@gmail.com, abo_hossam2011@hotmail.com, bilalsehole@gmail.com

ABSTRACT. I*n this paper, the problems of unsteady unidirectional flow of Maxwell fluid in a porous media are examined. The governing equations of flow are modelled, by employing the modified Darcy's law of a Maxwell fluid. Using Sumudu transform, analytical solutions of modelled equations are established for the following problems: (i) unsteady Couette flow, (ii) unsteady Poiseuille flow and (iii) unsteady generalized Couette flow. Since the Sumudu transform has units preserving properties, therefore aforementioned problems are solved without restoring the frequency domain. This is one of many strength points for this new transform, especially with respect to applications in problems with physical dimensions. Further, the solutions for the velocity fields that have been obtained; have complete agreement with those established by using the Laplace transform. Moreover, the corresponding solutions for Newtonian fluids as well as those for Maxwell fluids can be obtained as limiting cases of our solutions. Finally, the impact of relevant parameters on the velocity of fluids is also analyzed by graphical illustrations.*

Key Words: Sumudu transform, Maxwell fluid, Porous media, Velocity profile.

1. INTRODUCTION

The non-Newtonian fluids have been mainly classified under the differential, rate and integral types. The Maxwell fluids [1] are the subclass of non-Newtonian fluids and are the simplest subclass of rate type fluids which take the relaxation phenomena into consideration. It was employed to study numerous problems due to their relatively simple structure [2 - 4]. Moreover, one can reasonably hope to obtain analytical solutions from this type of Maxwell fluid. This thing motivates us to choose the Maxwell model in this study. The analytical solutions are important as these provide standard for checking the accuracies of many approximate solutions which can be numerical or empirical. They can also be used as tests for verifying numerical schemes that are developed for studying more complex flow problems.

All the above investigations of hydrodynamic fluids curbed the flows of Maxwell fluids in the non-porous medium [2 - 4]. Besides that the flow in porous media occurs widely in natural phenomena and in industrial applications such as oil recovery, food processing, building insulations, heat-storage beds, dispersion of chemical contaminants in various processes in the chemical industry and in the environment, to name just a few. Many of these applications involve non-Newtonian fluids in a porous media. But very little attention has been given to the flows of non-Newtonian fluids in porous media. Such investigations further narrow down when modified Darcy's law of non-Newtonian fluids have been taken into account. Most recently, some researches study the flow of non-Newtonian fluids in porous media [5 - 9] and references there in.

The aim of this communication is to examine the unsteady flows of a Maxwell fluid in a porous medium. The arrangement of the paper is as follows. In section 2, we document the governing equations. This is followed by the analysis of three transient flow problems between two parallel plates: (i) Couette flow (flow due to one of the plates starts suddenly and other being rest), (ii) Poiseuille flow (flow due to the constant pressure gradient between two fixed plates), (iii) generalized Couette flow (due to a constant pressure gradient between the plates one of them starts suddenly and other being rest).

To solve the aforementioned problems, we employ the Sumudu transform. It is an integral transform which was first time introduced by [10]. Sumudu transform can help to solve the complex applications in science and engineering, due to its simple formulation and consequent special and useful properties. Having scaling and units preserving properties, it may be used to solve problems without resorting to the frequency domain. This is one of many strength points for this new transform, especially with respect to applications in problems with physical dimensions. In fact, the Sumudu transform which is itself linear, preserves linear functions, and hence in particular does not change units [11, 12]. [13, 14] have shown it to be the theoretical dual to the Laplace transform, and hence ought to rival it in problem solving. This new transform was further developed and applied to many problems by various researchers [15 - 19] and references there in.

The Sumudu Transform is defined [10, 13] by

$$G(w)=S\left[f(t)\right](w)=\int_0^\infty f(wt)e^t dt, \quad \tau_1<w<\tau_2, \tag{1}$$

over the set of real functions

$$A=\left\{f(t)\,\middle|\,\exists M,\,\tau_1,\,\tau_2>0,\,\left|f(t)\right|<M\mathrm{e}^{\frac{|t|}{\tau_j}},\text{ if } t\in(-1)^j\times\left[0,\infty\right)\right\}.$$

In this work, we employ this technique to the unsteady unidirectional non-Newtonian fluid problems. All the expressions for velocity profiles are constructed for large and small times. To best our knowledge, Sumudu transform technique is first time applied to unsteady non-Newtonian fluid problems. It does not figure out in literature. Further, another new type of analytical solutions of the proposed problems is obtained, by using the composite function technique. At the end, several results of interest are obtained as the particular cases of the problems considered.

2. Governing equations

The continuity equation and the balance of linear momentum in a porous medium are

$$\nabla\cdot\mathbf{V}=0, \tag{2}$$

$$\rho\left[\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}\right]=\rho\mathbf{f}+div\mathbf{T}+\mathbf{R}, \tag{3}$$

where $\mathbf{V}$ is the velocity field, ρ is the density, $\mathbf{f}$ is the body force per unit mass, ∇ is the gradient operator, $\mathbf{R}$ is the Darcy's resistance and $\mathbf{T}$ is the Cauchy stress tensor in a Maxwell fluid satisfies the following expression

$$\mathbf{T}=-p\mathbf{I}+\mathbf{S};\;\; \mathbf{S}+\lambda\left(\frac{\partial\mathbf{S}}{\partial t}+(\nabla\cdot\mathbf{V})\mathbf{S}-\mathbf{LS}-\mathbf{SL}^T\right)=\mu\mathbf{A}_1, \tag{4}$$

in which p is hydrostatic pressure, $\mathbf{I}$ is the unit tensor, λ is the relaxation time, $\mathbf{S}$ is the extra-stress tensor, μ is the dynamic viscosity of the fluid, $\mathbf{L}$ is the velocity gradient $\mathbf{A}_1=\mathbf{L}+\mathbf{L}^T$, is the first Rivlin-Ericksen tensor and superscript T indicates the transpose operation.

According to Tan and Masuka [5], Darcy's resistance in an Oldroyd-B fluid satisfying the following expression:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\mathbf{R}=-\frac{\mu\phi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}, \tag{5}$$

where λ_r is the retardation time, ϕ is the porosity and k is the permeability of the porous medium. For Maxwell fluid $\lambda_r = 0$ and hence

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\mathbf{R} = -\frac{\mu\phi}{k}\mathbf{V}. \tag{6}$$

We select a velocity field of the following form

$$\mathbf{V} = (u(y,t),\, 0,\, 0), \tag{7}$$

where u is the velocity in the x direction. Further we assume that extra stress tensor is a function of y and t only i.e. $\mathbf{S} = \mathbf{S}(y,t)$. By using Eq. (7) into Eq. $(4)_2$, and remember that at $t = 0$ there is no motion of the fluid, we obtain $S_{xz} = S_{yy} = S_{yz} = S_{zz} = 0$, and

$$\left(1+\lambda\frac{\partial}{\partial t}\right)S_{xx} - 2\lambda S_{xy}\frac{\partial u}{\partial t} = 0,\quad \left(1+\lambda\frac{\partial}{\partial t}\right)S_{xy} - \mu\frac{\partial u}{\partial t} = 0. \tag{8}$$

Using Eq. (6) into Eq. (3), we have

$$\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial \mathbf{V}}{\partial t} = -\left(1+\lambda\frac{\partial}{\partial t}\right)\nabla p + \left(1+\lambda\frac{\partial}{\partial t}\right)div\mathbf{S} - \frac{\mu\phi}{k}\mathbf{V}, \tag{9}$$

In the absence of body forces Eqs. (7) - (9) gives

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t} = -\frac{1}{\rho}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - \beta u, \tag{10}$$

where $\nu = \mu/\rho$ is a kinematic viscosity and $\beta = \phi\nu/k$.

3. Unsteady Couette flow

Consider the flow between two infinite parallel rigid plates distance h apart. Both plates are initially at rest and the plate at $y = 0$ is fixed for all the time. The fluid motion starts suddenly due to constant velocity of the plate at $y = h$ in its own plane.

The governing equation with boundary and initial value problem is

$$\frac{\partial u}{\partial t} + \lambda\frac{\partial^2 u}{\partial t^2} = \nu\frac{\partial^2 u}{\partial y^2} - \beta u, \tag{11}$$

$$u(0,t) = 0, \quad \text{for all} \quad t,$$

$$u(h,t) = V, \quad \text{for} \quad t > 0, \tag{12}$$

$$\frac{\partial u(y,0)}{\partial t} = u(y,0) = 0, \quad \text{for} \quad 0 \le y < 0.$$

In this case, the solution is given by

$$\frac{u(y,t)}{V} = \frac{\sinh my}{\sinh mh} + 2\pi\sum_{n=1}^{\infty}\frac{(-1)^n n}{m^2h^2 + n^2\pi^2}\sin\left(\frac{n\pi}{h}y\right)T_n(t), \tag{13}$$

where

$$T_n(t) = e^{-a_1 t}\left[\cosh\sqrt{a_1^2 - d_1^2 t} + \frac{a_1}{\sqrt{a_1^2 - d_1^2 t}}\sinh\sqrt{a_1^2 - d_1^2 t}\right],\ a_1 > d_1,$$

$$T_n(t) = e^{-a_1 t}(1 - a_1),\quad a_1 = d_1,$$

$$T_n(t) = e^{-a_1 t}\left[\cos\sqrt{d_1^2 - a_1^2 t} + \frac{a_1}{\sqrt{d_1^2 - a_1^2 t}}\sin\sqrt{d_1^2 - a_1^2 t}\right],\quad a_1 < d_1,$$

$$m = \left(\frac{\beta}{\nu}\right)^{1/2},\quad d_1 = \left(\frac{\beta + \nu\left(\frac{n\pi}{h}\right)^2}{\lambda}\right)^{1/2},\quad a_1 = \frac{1}{2\lambda}.$$

If the upper plate is fixed and the lower one moves at a constant Speed V, then the velocity distribution is

$$\frac{u(y,t)}{V} = \frac{\sinh m(h-y)}{\sinh mh} + 2\pi\sum_{n=1}^{\infty}\frac{(-1)^n n}{m^2h^2 + n^2\pi^2}\sin\left(\frac{n\pi}{h}y\right)T_n(t).$$

3.1. Solution for small time

In this section we find the solution by Sumudu transform technique, defined by Eq. (1). The Sumudu transform of Eq. (11) and the boundary conditions (12) take the following forms

$$\frac{d^2G(y,w)}{dy^2} - s^2G(y,w) = 0, \tag{14}$$

$$G(0,w) = 0,\quad G(h,w) = V, \tag{15}$$

where $s^2 = \left(\frac{\lambda + w + \beta w^2}{\nu w^2}\right)$.

The solution of Eq. (14) by using boundary conditions (15) can be written as

$$\frac{G(y,w)}{V} = \frac{\sinh sy}{\sinh sh}. \tag{16}$$

The inverse sumudu transform is defined by [12] in of the form

$$S^{-1}[G(w)] = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{wt}G\left(\frac{1}{w}\right)\frac{dw}{w} = \sum residues\left[e^{wt}\frac{G\left(\frac{1}{w}\right)}{w}\right]. \tag{17}$$

Sumudu inversion of Eq. (16) yields

$$\frac{u(y,t)}{V} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{\sinh\left(\frac{\lambda w^2 + w + \beta}{\nu}\right)y}{w\sinh\left(\frac{\lambda w^2 + w + \beta}{\nu}\right)h}e^{wt}dw. \tag{18}$$

In Eq. (18), $w = 0$ is a simple pole. Therefore residue at $w = 0$ is

$$Res(0) = \frac{\sinh my}{\sinh mh},\ \text{where}\ m = \sqrt{\frac{\beta}{\nu}}. \tag{19}$$

The other singular points are the zeros of

$$\sinh\left(\frac{\lambda w^2 + w + \beta}{\nu}\right)h = 0.\ \text{Setting}\ \sqrt{\frac{\lambda w^2 + w + \beta}{\nu}} = i\alpha, \tag{20}$$

we find that

$$\sinh\alpha h = 0, \tag{21}$$

and $\alpha_n = n\pi/h,\ n = 1, 2, 3, \ldots, \infty$ are the zeros of Eq. (21), then

$$w_{1n}, = \frac{-1 + \sqrt{1 - 4\lambda(\beta + \nu\alpha_n^2)}}{2\lambda},\quad w_{2n} = \frac{-1 - \sqrt{1 - 4\lambda(\beta + \nu\alpha_n^2)}}{2\lambda},$$

$n = 1, 2, 3, \ldots, \infty$ are the poles. Since all α_ns are symmetrically placed about origin on the real axis, all poles (w_{1n}, w_{2n}) lies on the negative real axis. These are the simple poles and the residue at all these poles can be obtained as

$$Res(w_{1n}) = \left(-\frac{2\pi\nu}{h^2}\right)\frac{(-1)^n ne^{w_{1n}t}}{w_{1n}(1 + 2\lambda w_{1n})}\sin\left(\frac{n\pi}{h}y\right),$$

$$Res(w_{1n}) = \left(-\frac{2\pi\nu}{h^2}\right)\frac{(-1)^n ne^{w_{2n}t}}{w_{2n}(1 + 2\lambda w_{2n})}\sin\left(\frac{n\pi}{h}y\right).$$

Adding $Res(0), Res(w_{1n})$ and $Res(w_{1n})$, a complete solution is obtained as

$$\frac{u(y,t)}{V} = \frac{\sinh my}{\sinh mh} - \left(\frac{2\pi\nu}{h^2}\right)\sum_{n=1}^{\infty}(-1)^n n \left[\frac{e^{w_{1n}t}}{w_{1n}(1 + 2\lambda w_{1n})} + \frac{e^{w_{2n}t}}{w_{2n}(1 + 2\lambda w_{2n})}\right]\sin\left(\frac{n\pi}{h}y\right). \tag{22}$$

The solution obtained in Eq. (22) is identical to those given by Laplace transform method taking $\theta = 0$, and $N = 0$ [Eq. 25, 20].

In addition, let us give another expression for the velocity field $u(y,t)$, for this we rewrite Eq. (18) in the form of composite function

$$G(y,\,q)=\frac{V}{q}A(y,\,q). \tag{23}$$

We consider the functions

$$F(y,\,q)=\frac{\sinh\left(\frac{y}{\sqrt{\nu}}\sqrt{q}\right)}{\sinh\left(\frac{h}{\sqrt{\nu}}\sqrt{q}\right)},\quad p(q)=\lambda q^2+q+\beta. \tag{24}$$

Now we have

$$A(y,\,q)=(F\circ p)(q)=\frac{\sinh\left(\frac{y}{\sqrt{\nu}}\sqrt{p(q)}\right)}{\sinh\left(\frac{h}{\sqrt{\nu}}\sqrt{p(q)}\right)}. \tag{25}$$

We find the inverse Sumudu transform of the function $F(y,q)$ by means of the residue theorem for this

$$\sinh\left(\frac{h}{\sqrt{\nu}}\sqrt{p(q)}\right)=0,\qquad q_k=-\frac{\nu k^2\pi^2}{h^2},\ k=1,\,2,\,3,\,\ldots,\,\infty$$

$$\Rightarrow\ f(y,\,t)=S^{-1}[F(y,\,q)]=\frac{2\nu\pi}{h^2}\sum_{k=1}^{\infty}(-1)^{k+1}k\sin\left(\frac{k\pi y}{h}\right)\mathrm{e}^{-\frac{\nu k^2\pi^2}{h^2}t}. \tag{26}$$

And

$$a(y,\,t)=S^{-1}[A(y,\,q)]=S^{-1}[(F\circ p)(q)]=\int_0^{\infty}f(y,\,z)g(z,\,t)dz, \tag{27}$$

where $g(z,\,t)=S^{-1}[\mathrm{e}^{-zp(q)}]$.

Using the values of $g(z,\,t)$ and $f(y,\,z)$ in Eq. (27), we get

$$a(y,\,t)=\frac{\nu\pi t}{h^2}\mathrm{e}^{-\frac{t}{2\lambda}}\sum_{k=1}^{\infty}(-1)^{k+1}k\sin\left(\frac{k\pi y}{h}\right)$$
$$\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{(n+1)!(2n+1)!}\int_0^{\infty}S^{2n+1}J_2\left(2\sqrt{st}\right)\int_0^{\infty}z^n\mathrm{e}^{-z\left(\frac{\nu k^2\pi^2}{h^2}+\beta-\frac{1}{4\lambda}\right)}dz. \tag{28}$$

Using the expressions (A1) - (A4) from appendix and after simplification Eq. (28) becomes

$$a(y,\,t)=\frac{2\nu\pi}{h^2\sqrt{\lambda}}\mathrm{e}^{-\frac{t}{2\lambda}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{\sqrt{a_k}}\sin\left(\frac{k\pi y}{h}\right)\sin\left(\sqrt{\frac{a_k}{\lambda}}t\right). \tag{29}$$

From Eq. (23), we have

$$u(y,\,t)=V\int_0^t a(y,\,s)ds. \tag{30}$$

Using Eq. (29) into the above Eq. (30) and after solving integration by parts, we obtain a new expression for velocity

$$\frac{u(y,\,t)}{V}=2\nu\pi\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{\nu k^2\pi^2+h^2\beta}\sin\left(\frac{k\pi y}{h}\right)-2\nu\pi\mathrm{e}^{-\frac{t}{2\lambda}}$$
$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{\nu k^2\pi^2+h^2\beta}\sin\left(\frac{k\pi y}{h}\right)\left[\cos\left(\sqrt{\frac{a_k}{\lambda}}t\right)+\frac{1}{2\sqrt{\lambda a_k}}\sin\left(\sqrt{\frac{a_k}{\lambda}}t\right)\right]. \tag{31}$$

4. Unsteady Poiseuille flow

In this section, we discuss another type of unsteady flow situation, that the fluid between two parallel plates which are stationary is set in motion due to sudden application of a constant pressure gradient is termed as the poiseuille flow. Suppose that the fluid is bounded by two parallel plates at $y=\pm h,$ and it is initially at rest and fluid starts suddenly due to a constant pressure gradient.

The governing equations is (10), and the initial and boundary conditions are

$$u(\pm h,\,t)=0,\quad \text{for all}\quad t,$$
$$\frac{\partial u(y,\,t)}{\partial t}=u(y,\,t)=0,\quad \text{for}\quad h\le y\le h, \tag{32}$$

where $2h$ is the distance between two parallel plates.

It then follows that a solution of the following form exists:

$$\frac{u(y,\,t)}{-\left(\frac{1}{\beta\rho}\right)\frac{dp}{dx}}=1-\frac{\cosh my}{\cosh mh}-\frac{16m^2h^2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}$$
$$\left(\frac{1}{4m^2h^2+(2n+1)\pi^2}\right)\cos\left(\frac{(2n+1)\pi}{2h}y\right)T_n(t), \tag{33}$$

where

$$T_n(t)=\mathrm{e}^{-a_1t}\left[\cosh\sqrt{a_1^{\,2}-d_1^{\,2}t}+\frac{a_1}{\sqrt{a_1^{\,2}-d_1^{\,2}t}}\sinh\sqrt{a_1^{\,2}-d_1^{\,2}t}\right],\ a_1>d_1,$$

$$T_n(t)=\mathrm{e}^{-a_1t}(1-a_1),\quad a_1=d_1,$$

$$T_n(t)=\mathrm{e}^{-a_1t}\left[\cos\sqrt{d_1^{\,2}-a_1^{\,2}t}+\frac{a_1}{\sqrt{d_1^{\,2}-a_1^{\,2}t}}\sin\sqrt{d_1^{\,2}-a_1^{\,2}t}\right],\quad a_1<d_1,$$

$$d_1=\left(\frac{\beta+\nu\left(\frac{(2n+1)\pi}{2h}\right)^2}{\lambda}\right)^{1/2},\quad a_1=\frac{1}{2\lambda},\quad m=\left(\frac{\beta}{\nu}\right)^{1/2}.$$

4.1. Solution for small time

After taking Sumudu transform, Eq. (10) and boundary conditions (32) give

$$\frac{d^2G(y,\,w)}{dy^2}-s^2G(y,\,w)=\frac{1}{\mu}\frac{dp}{dx}, \tag{34}$$

$$G(+h,\,w)=0,\quad G(-h,\,w)=0, \tag{35}$$

The solution of Eq. (38) by using boundary conditions (35) can be written as

$$\frac{-\mu G(y,\,w)}{\frac{dp}{dx}}=\frac{1}{s^2}\left[1-\frac{\cosh sy}{\cosh sh}\right]. \tag{36}$$

The inverse Sumudu transform of Eq. (36) is

$$\frac{u(y,\,t)}{V}=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{\mathrm{e}^{wt}}{w(\lambda w^2+w+\beta)}\left[1-\frac{\cosh sy}{\cosh sh}\right]dw. \tag{37}$$

For the inverse solution of Eq. (37), we employ the similar procedure as in the first half of section 3.1. In order to avoid the detail, the solution is given by

$$\frac{u(y,\,t)}{-\left(\frac{1}{\beta\rho}\right)\frac{dp}{dx}}=1-\frac{\cosh my}{\cosh mh}+\frac{\beta\mathrm{e}^{w_1t}}{w_1(w_1-w_2)}\left[1-\frac{\cosh s_1y}{\cosh s_1h}\right]$$
$$+\frac{\beta\mathrm{e}^{w_2t}}{w_2(w_2-w_1)}\left[1-\frac{\cosh s_2y}{\cosh s_2h}\right]-\left(\frac{\beta\nu\pi}{h^2}\right)\sum_{n=0}^{\infty}(-1)^k(2k+1)$$
$$\left[\frac{\mathrm{e}^{w_{1k}t}}{w_{1k}(\beta+d_1w_{1k}+d_2w_{1k}^{\;2}+d_3w_{1k}^{\;3})}+\frac{\mathrm{e}^{w_{2k}t}}{w_{2k}(\beta+d_1w_{2k}+d_2w_{2k}^{\;2}+d_3w_{2k}^{\;3})}\right]\cos\left(\frac{(2k+1)\pi}{2h}y\right), \tag{38}$$

where

$$w_1=\frac{-1+\sqrt{1-4\lambda\beta}}{2\lambda},\quad w_2=\frac{-1-\sqrt{1-4\lambda\beta}}{2\lambda}$$

$$s_1=\sqrt{\frac{\lambda w_1^{\,2}+w_1+\beta}{\nu}},\quad s_2=\sqrt{\frac{\lambda w_2^{\,2}+w_2+\beta}{\nu}}$$

$$d_1=(1+2\lambda\beta),\quad d_2=2\lambda,\quad d_3=2\lambda^2.$$

The solution obtained in Eq. (38) is identical to those given by Laplace transform method taking $\theta=0,$ and $N=0$ in [Eq. (42), 20].

For another expression of the velocity $u(y,t),$ we rewrite Eq. (37) in the form of composite function

$$\frac{G(y,q)}{-\frac{1}{\rho}\frac{dp}{dx}}=\frac{1}{q(\lambda q^2+q+\beta)}A(y,q). \tag{39}$$

We consider the functions

$$F(y,q)=\frac{\cosh\left(\frac{y}{\sqrt{\nu}}\sqrt{q}\right)}{\cosh\left(\frac{h}{\sqrt{\nu}}\sqrt{q}\right)},\quad p(q)=\lambda q^2+q+\beta. \tag{40}$$

Now we have

$$A(y,q)=(F\circ p)(q)=\frac{\sinh\left(\frac{y}{\sqrt{\nu}}\sqrt{p(q)}\right)}{\sinh\left(\frac{h}{\sqrt{\nu}}\sqrt{p(q)}\right)}. \tag{41}$$

Now, we find the inverse Sumudu transform of the function $F(y,q)$ by means of the residue theorem for this

$$\sinh\left(\frac{h}{\sqrt{\nu}}\sqrt{p(q)}\right)=0,\qquad q_k=-\frac{\nu}{h^2}\left[\frac{(2k+1)\pi}{2}\right]^2,\ k=1,2,3,\ldots,\infty$$

$$\Rightarrow\ f(y,t)=S^{-1}[F(y,q)]=\frac{\nu\pi}{h^2}\sum_{k=1}^{\infty}(-1)^k(2k+1)\cos\left(\frac{(2k+1)\pi y}{2h}\right)e^{-q_k t}. \tag{42}$$

$$a(y,t)=S^{-1}[A(y,q)]=S^{-1}[(F\circ p)(q)]=\int_0^{\infty}f(y,z)g(z,t)dz. \tag{43}$$

Using the values of $g(z,t)$ and $f(y,z)$ in Eq. (43), we get

$$a(y,t)=\frac{\nu\pi t}{2h^2}e^{-\frac{t}{2\lambda}}\sum_{k=1}^{\infty}(-1)^k(2k+1)\cos\left(\frac{(2k+1)\pi y}{2h}\right)\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{(n+1)!(2n+1)!}\int_0^{\infty}S^{2n+1}J_2\left(2\sqrt{st}\right)\int_0^{\infty}z^n e^{-za_k}dz, \tag{44}$$

where $a_k=\frac{\nu}{h^2}\left(\frac{(2k+1)\pi}{2}\right)^2+\beta-\frac{1}{4\lambda}.$

Using the same expressions (A_1) - (A_4) from appendix and after simplification we get

$$a(y,t)=\frac{\nu\pi}{h^2\sqrt{\lambda}}e^{-\frac{t}{2\lambda}}\sum_{k=1}^{\infty}\frac{(-1)^k(2k+1)}{\sqrt{a_k}}\cos\left(\frac{(2k+1)\pi y}{2h}\right)\sin\left(\sqrt{\frac{a_k}{\lambda}}t\right). \tag{45}$$

Now from Eq. (39), we let

$$B(q)=\frac{1}{q(\lambda q^2+q+\beta)}.$$

Solving $B(q)$ by partial fraction, we have

$$b(t)=S^{-1}\left[B(q)\right]=\frac{1}{\beta}-\frac{1}{\beta}e^{-\frac{t}{2\lambda}}\cosh\left(\frac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)-\frac{1}{\beta\sqrt{1-4\lambda\beta}}e^{-\frac{t}{2\lambda}}\sinh\left(\frac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right). \tag{46}$$

From Eq. (39), we have

$$\frac{G(y,q)}{-\frac{1}{\rho}\frac{dp}{dx}}=B(q)-B(q)\bullet A(y,q),$$

Using the convolution theorem $S^{-1}[B(q),A(y,q)]=(b*a)(t)=\int_0^t a(s)b(t-s)ds=\int_0^t a(t-s)b(s)ds,$ we obtain a new expression for the velocity field $u(y,t)$

$$\frac{u(y,t)}{-\left(\frac{1}{\beta\rho}\right)\frac{dp}{dx}}=1-e^{-\frac{t}{2\lambda}}\cosh\left(\frac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)-\frac{1}{\sqrt{1-4\lambda\beta}}e^{-\frac{t}{2\lambda}}\sinh\left(\frac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)-\frac{4\nu\lambda\pi}{h^2}\sum_{k=1}^{\infty}\frac{(-1)^k(2k+1)}{1+4\lambda a_k}\cos\left(\frac{(2k+1)\pi y}{2h}\right)$$
$$+\frac{2\nu\sqrt{\lambda}\pi}{h^2}e^{-\frac{t}{2\lambda}}\sum_{k=1}^{\infty}\frac{(-1)^k(2k+1)}{\sqrt{a_k}(1+4\lambda a_k)}\cos\left(\frac{(2k+1)\pi y}{2h}\right)\left[\sin\left(\sqrt{\frac{a_k}{\lambda}}t\right)+2\sqrt{\lambda a_k}\cos\left(\sqrt{\frac{a_k}{\lambda}}t\right)\right]+\frac{4}{\pi}e^{-\frac{t}{2\lambda}}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)}\cos\left(\frac{(2k+1)\pi y}{2h}\right)$$
$$\left[\begin{array}{l}\cosh\left(\frac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)+\frac{1}{\sqrt{1-4\lambda\beta}}\sinh\left(\frac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)\\ -\cos\left(\sqrt{\frac{a_k}{\lambda}}t\right)-\frac{1}{2\sqrt{\lambda a_k}}\sin\left(\sqrt{\frac{a_k}{\lambda}}t\right)\end{array}\right]. \tag{47}$$

5. Unsteady generalized Couette flow

Suppose the fluid is bounded by two parallel plates at $y=0$ and $y=h,$ and it is initially at rest. The fluid starts suddenly due to a pressure gradient and by the motion of the upper plate. The governing equations are (10), and the initial and boundary conditions are:

$$u(0,t)=0,\quad \text{for all}\quad t,$$
$$u(h,t)=V,\quad \text{for}\quad t>0, \tag{48}$$
$$\frac{\partial u(y,0)}{\partial t}=u(y,0)=0,\quad \text{for}\quad 0\le y<0.$$

Employing the similar procedure of section 3, we have

$$\frac{u(y,t)}{V}=\frac{\sinh my}{\sinh mh}+\Gamma\left(1-\frac{\sinh my}{\sinh mh}-\frac{\sinh m(h-y)}{\sinh mh}\right)-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{n^2\pi^2+m^2h^2\Gamma(1-(-1)^n)}{m^2h^2+n^2\pi^2}\right)\sin\left(\frac{n\pi y}{h}\right)T_n(t), \tag{49}$$

where, $\Gamma=-\left(dp/dx\right)/\left(\rho\beta V\right).$

If the upper plate is held fixed and the lower plate move at a speed $V,$ then the velocity distribution becomes

$$\frac{u(y,t)}{V}=\frac{\sinh m(h-y)}{\sinh mh}+\Gamma\left(1-\frac{\sinh my}{\sinh mh}-\frac{\sinh m(h-y)}{\sinh mh}\right)-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{n^2\pi^2+m^2h^2\Gamma(1-(-1)^n)}{m^2h^2+n^2\pi^2}\right)\sin\left(\frac{n\pi y}{h}\right)T_n(t),$$

5.1. Solution for small time

In this section, after taking Sumudu transform Eq. (10) becomes Eq. (34) and initial-boundary conditions (48) becomes (15) respectively, and adopting the same method of solution in section 4.1, we arrive at

$$\frac{u(y,t)}{V}=\frac{\sinh my}{\sinh mh}+\Gamma\left(1-\frac{\sinh my}{\sinh mh}-\frac{\sinh m(h-y)}{\sinh mh}\right)$$
$$+\beta\Gamma\left[\begin{array}{l}\dfrac{e^{w_1 t}}{w_1(w_1-w_2)}\left(1-\dfrac{\sinh s_1 y}{\sinh s_1 h}-\dfrac{\sinh s_1(h-y)}{\sinh s_1 h}\right)+\\ \dfrac{e^{w_2 t}}{w_2(w_2-w_1)}\left(1-\dfrac{\sinh s_2 y}{\sinh s_2 h}-\dfrac{\sinh s_2(h-y)}{\sinh s_2 h}\right)\end{array}\right] \tag{50}$$
$$-\left(\frac{2\nu\pi}{h^2}\right)\sum_{n=1}^{\infty}(-1)^n n\left[\frac{e^{w_{1n}t}}{w_{1n}(1+2\lambda w_{1n})}+\frac{e^{w_{2n}t}}{w_{2n}(1+2\lambda w_{2n})}\right]\sin\left(\frac{n\pi}{h}y\right)$$
$$+\frac{\nu\beta\pi\Gamma}{h^2}\sum_{n=1}^{\infty}(-1)^n n\left[\begin{array}{l}\dfrac{e^{w_{1k}t}}{w_{1k}(\beta+d_1 w_{1k}+d_2 w_{1k}{}^2+d_3 w_{1k}{}^3)}+\\ \dfrac{e^{w_{2k}t}}{w_{2k}(\beta+d_1 w_{2k}+d_2 w_{2k}{}^2+d_3 w_{2k}{}^3)}\end{array}\right]$$
$$\left[\sin\left(\frac{n\pi}{h}y\right)+\sin\left(\frac{n\pi}{h}(h-y)\right)\right].$$

The solution obtained in Eq. (50) is identical to those given by Laplace transform method taking $\theta=0,$ and $N=0$ in [Eq. (48), 20].

For another expression of the velocity field, we adopt the same procedure as in the second half of section of 4.1, and we get the following new expression of the velocity field

$$\frac{u(y,t)}{V}=\Gamma\left[\begin{array}{l}\dfrac{1}{\beta}-\dfrac{1}{\beta}e^{-\frac{t}{2\lambda}}\cosh\left(\dfrac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)\\ -\dfrac{1}{\beta\sqrt{1-4\lambda\beta}}e^{-\frac{t}{2\lambda}}\sinh\left(\dfrac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)\end{array}\right]$$
$$+\frac{8\nu\lambda\pi}{h^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{1+4\lambda a_k}\sin\left(\frac{k\pi y}{h}\right)-\frac{4\nu\sqrt{\lambda}\pi}{h^2}e^{-\frac{t}{2\lambda}}$$
$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{\sqrt{a_k}(1+4\lambda a_k)}\sin\left(\frac{k\pi y}{h}\right)\left[\sin\left(\sqrt{\frac{a_k}{\lambda}}t\right)+2\sqrt{\lambda a_k}\cos\left(\sqrt{\frac{a_k}{\lambda}}t\right)\right]-$$
$$-\Gamma\left[\begin{array}{l}\dfrac{8\nu\lambda\pi}{h^2}\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{1+4\lambda a_k}\sin\left(\frac{k\pi y}{h}\right)-\frac{4\nu\sqrt{\lambda}\pi}{h^2}e^{-\frac{t}{2\lambda}}\\ \displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{\sqrt{a_k}(1+4\lambda a_k)}\sin\left(\frac{k\pi y}{h}\right)\\ \left\{\sin\left(\sqrt{\dfrac{a_k}{\lambda}}t\right)+2\sqrt{\lambda a_k}\cos\left(\sqrt{\dfrac{a_k}{\lambda}}t\right)\right\}-\\ \dfrac{2}{\beta\pi}e^{-\frac{t}{2\lambda}}\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\left(\frac{k\pi y}{h}\right)\left\{\begin{array}{l}\cosh\left(\dfrac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)+\dfrac{1}{\sqrt{1-4\lambda\beta}}\\ \sinh\left(\dfrac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)-\cos\left(\sqrt{\dfrac{a_k}{\lambda}}t\right)\\ -\dfrac{1}{2\sqrt{\lambda a_k}}\sin\left(\sqrt{\dfrac{a_k}{\lambda}}t\right)\end{array}\right\}\end{array}\right]$$

$$-\Gamma\left[\begin{array}{l}\dfrac{8\nu\lambda\pi}{h^2}\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{1+4\lambda a_k}\sin\left(\frac{k\pi}{h}(h-y)\right)-\frac{4\nu\sqrt{\lambda}\pi}{h^2}e^{-\frac{t}{2\lambda}}\\ \displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{\sqrt{a_k}(1+4\lambda a_k)}\sin\left(\frac{k\pi}{h}(h-y)\right)\\ \left\{\sin\left(\sqrt{\dfrac{a_k}{\lambda}}t\right)+2\sqrt{\lambda a_k}\cos\left(\sqrt{\dfrac{a_k}{\lambda}}t\right)\right\}-\dfrac{2}{\beta\pi}e^{-\frac{t}{2\lambda}}\\ \displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\left(\frac{k\pi}{h}(h-y)\right)\left\{\begin{array}{l}\cosh\left(\dfrac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)+\dfrac{1}{\sqrt{1-4\lambda\beta}}\\ \sinh\left(\dfrac{\sqrt{1-4\lambda\beta}}{2\lambda}t\right)-\cos\left(\sqrt{\dfrac{a_k}{\lambda}}t\right)\\ -\dfrac{1}{2\sqrt{\lambda a_k}}\sin\left(\sqrt{\dfrac{a_k}{\lambda}}t\right)\end{array}\right\}\end{array}\right] \tag{51}$$

6. Special Cases

The Maxwell fluid in porous media is the general case of the Newtonian fluid. When $\lambda=0,$ and $\beta=0,$ it reduce to the Newtonian fluid and when $\lambda=0$ it gives the results of Newtonian fluid in Porous media. Also when $\beta=0$ it gives the results of Maxwell fluid without porous media.

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8. CONCLUSIONS AND NUMERICAL RESULTS

In this work, we have constructed the exact analytical solutions of the three flow problems of a unsteady unidirectional Maxwell fluid between two parallel plates. The presented analysis is valid for large and small times. To obtain the analytical solutions, we have successfully applied the Sumudu Transform to the proposed problems. The obtained results are valid for small times and match with those obtained by Laplace transform in Literature [20]. But, it shows that the Sumudu transform is much simpler and powerful tool than Laplace transform to solve the engineering and physical problems due to having units preserving properties and without resorting the frequency domain. Further, by using the compound function method new exact solutions are constructed. Finally, the special cases of the problems are discussed.

In order to discuss some physical aspects of the Couette flows, the figures 1 and 2 have been plotted. Fig. 1 contains graphs of the velocity $u(y,t)$ corresponding to Maxwell and Newtonian fluids for the porosity parameter $\beta=0,$ and $\beta\neq 0.$ These diagrams are plotted verses the spatial coordinate $y\in[0,h],$ $h=0.5,$ $\nu=0.012$ and for V=0.75, $\lambda=1.75,$ and several values of the time t. For small values of time t, the velocity of Maxwell fluid is equal to zero near the bottom plate. Obviously, in the case when porosity parameter is zero the fluid velocity is bigger than in the $\beta\neq 0.$ For large values of time *t* the velocity profiles for Maxwell and Newtonian fluids become identically. Fig. 2 shows the influence of relaxation time λ on the velocity field $u(y,t)$. For small values of time *t*, the Maxwell fluid characterized by a bigger relaxation time has smaller velocity. This property is changed if the values of time *t* increase. Also, for large values of time t, the velocity profiles become identically.

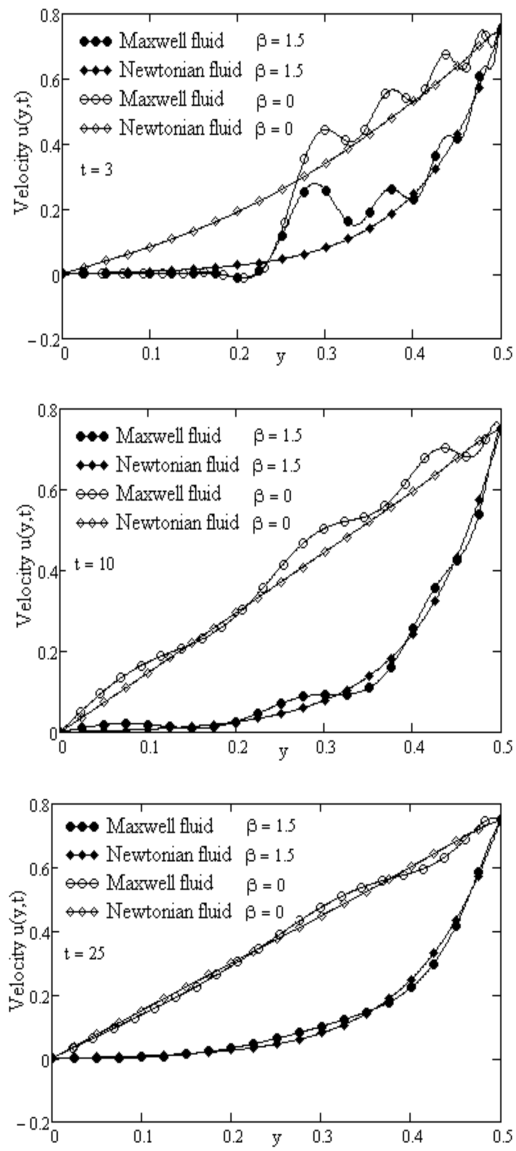
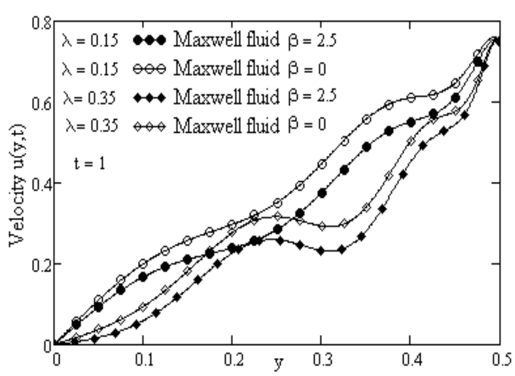


Fig.1. Velocity profiles for Couette flows of Maxwell and Newtonian fluids (V = 0.75, λ = 1.75, ν = 0.012)



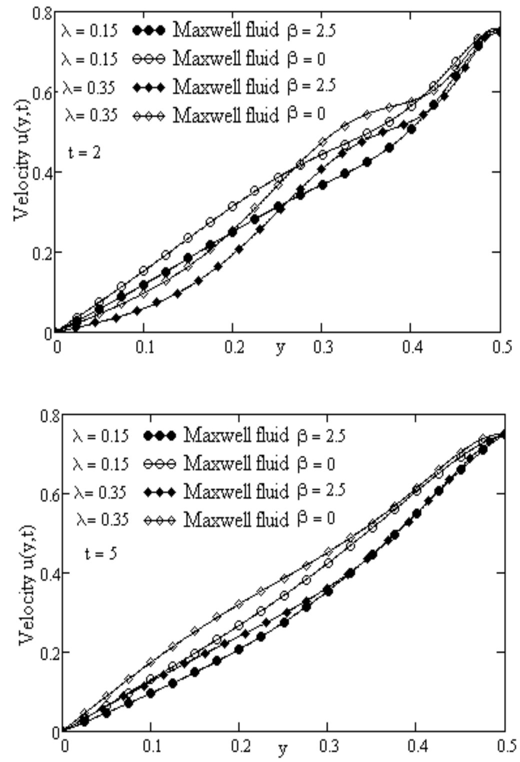


Fig. 2. Velocity profiles for Couette flows of Maxwell fluids (V = 0.75, ν = 0.35)

9. Appendix

$$\int_0^\infty z^n e^{-z\left(\frac{\nu k^2\pi^2}{h^2}+\beta-\frac{1}{4\lambda}\right)}dz=\frac{n!}{a_k^{\,n+1}},\quad \text{where}\quad a_k=\frac{\nu k^2\pi^2}{h^2}+\beta-\frac{1}{4\lambda}>0, \tag{A$_1$}$$

$$\sum_{n=0}^{\infty}\frac{(-\lambda)^n S^{2n+1}}{(n+1)!(2n+1)!}=\frac{2}{\lambda s}\left(1-\cos\left(\frac{\sqrt{\lambda s}}{\sqrt{a_k}}\right)\right). \tag{A$_2$}$$

$$\int_0^\infty \frac{1}{s}J_2\left(2\sqrt{st}\right)ds=1, \tag{A$_3$}$$

$$\int_0^\infty \frac{1}{s}\cos\left(\frac{\sqrt{\lambda s}}{\sqrt{a_k}}\right)J_2\left(2\sqrt{st}\right)ds=\int_0^\infty \frac{1}{x}\cos\left(C_k x^2\right)J_2(d)dx, \tag{A$_4$}$$

where $x=2\sqrt{st}$, $C_k=1/4t\sqrt{\lambda/a_k}$.

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